Flag Varieties and Representation Theory

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Introduction

The outline of these notes are as follows.

1. Representation theory of $S_n$ and $GL_n$ - Character theory, Specht modules, symmetric functions, Schur-Weyl duality
2. Representation theory of Lie groups and Lie algebras
3. Algebraic groups and reflection groups – Root systems, Coxeter combinatorics
4. Flag varieties, Grassmannians, intersection theory

For most of Chapter 1, these notes are drawn from Fulton and Harris. We use Sagan for the tabloid approach to representations of $S_n$. For Schur-Weyl duality, L. is used.
Part I

Representation theory of $S_n$ and $GL_n$
Chapter 1

Tensor products

1.1 Definitions

Let $V$ and $W$ be vector spaces with bases $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_m\}$ respectively. Then the tensor product $V \otimes W$ is a vector space with dimension $mn$, consisting of basis elements $\{v_i \otimes w_j\}$.

Alternatively, for any two vector spaces, $V \otimes W$ is the vector space consisting of $V \star W$ subject to the relations:

$$
(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w
$$
$$
v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2
$$
$$
c(v \otimes w) = cv \otimes w = v \otimes cw
$$

We can use the second definition to check that the tensor product satisfies the universal property. This is an abstract concept that can actually clean up some proofs fairly nicely. For completeness, we’ll specify the field with $\otimes_k$ (this is implied in the previous definitions).

The tensor product of $k$-vector spaces $V$ and $W$ is another vector space $V \otimes_k W$ with a $k$-bilinear map $\tau: V \times W \to V \otimes_k W$ such that for every $k$-bilinear map $\phi: V \times W \to X$, there is a unique linear map $\psi: V \otimes_k W \to X$ such that $\phi = \psi \circ \tau$.

As stated earlier, it can be shown that the construction described above satisfies the universal property. The universal property can be used, among other things, to show that the tensor product is unique up to isomorphism. (Left as exercises!)

1.2 Symmetric and alternating powers

If we take the tensor product of a vector space $V$ with itself a bunch of times, we get an interesting object $V \otimes^n$. Understanding the structure of $V \otimes^n$, can be surprisingly nontrivial and interesting. To start, there are two subspaces of importance: the symmetric power and the exterior power. Given $T \in V \otimes^n$, let $s(T)$ be one of the $\binom{n}{2}$ transpositions that switches two of the coordinates of each element making up $T$. 
Definition 1.2.1. The symmetric power, $S^nV$, is the quotient space of $V^\otimes n$ by tensors of the form $T - s(T)$.

Definition 1.2.2. The exterior power, $\wedge^n V$, is the quotient space of $V^\otimes n$ by tensors $T$ such that $T = s(T)$.

These definitions may be a bit abstract, but what they are saying is that if $T$ is a symmetric tensor, then $T = s(T)$ for all transpositions $s$. If $T$ is an alternating tensor, then $T = -s(T)$. For instance, $v_1 \otimes v_2 + v_2 \otimes v_1 \in S^2 V$ and $v_1 \otimes v_2 - v_2 \otimes v_1 \in \wedge^2 V$.

It turns out that we can also define the tensor product of linear maps. It is defined very naturally: if $A : V \to V'$ and $B : W \to W'$ are linear maps, then $A \otimes B : V \otimes W \to V' \otimes W'$ is a linear map defined by $A \otimes B(v \otimes w) = Av \otimes Bw$ and extending linearly. Naturally, this also defines maps $S^n A : S^n V \to S^n V'$ and $\wedge^n A : \wedge^n V \to \wedge^n V'$.

Now let us return to $V^\otimes n$ where $n = 2$. We claim that $V^\otimes 2 = S^2 V \oplus \wedge^2 V$. Indeed, there from any tensor $T = \sum v_i \otimes v_j \in V^\otimes 2$, we can construct a symmetric tensor $S(T) = \sum v_i \otimes v_j + v_j \otimes v_i$. Similarly, we can construct an alternating tensor $A(T) = \sum v_i \otimes v_j - v_j \otimes v_i$. Then every tensor can be written uniquely as the sum of two elements from these two non disjoint subspaces, proving the desired direct sum decomposition.

1.3 Hilbert’s third problem

Hilbert’s third problem can be stated very simply. It was actually proposed and solved as part of a math contest in the 1880s, but the solution we present here is due to Dehn and is very slick.

Problem 1.3.1. Given two polyhedra of equal volume, can we always cut the first with straight lines into finitely many pieces and rearrange them to make the second?

Solution 1.3.2. The answer is no. We show this by defining the Dehn invariant for a polyhedron. Given a polyhedron $A$, let $D(A) = \sum a l(a) \otimes \frac{\beta(a)}{\pi} \in \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q}$, where $a$ ranges over the edges of $A$, $l(a)$ is the length of $a$, and $\beta(a)$ is the dihedral angle at $a$. Cutting with a straight line preserves $D(A)$.

Now it suffices to show that the regular tetrahedron and the cube have different Dehn invariants. Take a cube with side length 1. Its Dehn invariant is 0. The tetrahedron will have edge length $\sqrt{72}$, and the dihedral angle $\theta$ satisfies $\cos \theta = \frac{1}{3}$. Now it suffices to show that $\theta$ is not a rational multiple of $\pi$. But it is well-known (through algebraic numbers) that the only the only possible rational values are $\{0, \pm 1, \pm \frac{1}{2}\}$.

1.4 A glimpse of Schur functors

It turns out that the symmetric and alternating powers can be naturally generalized. One way to look for such a generalization is to note that $V^\otimes 2$ decomposes as $S^2 V \oplus \wedge^2 V$ (these are irreducible, as we will see later), and try to generalize this to $V^\otimes n$. The irreducible modules we get this way are...
called Schur functors. In this subsection we will simply explain where they come from, and then come back to them in more depth when we discuss Schur-Weyl duality.

A **partition** of $n$ is a set of positive integers that sum to $n$, usually written in decreasing order. Now consider the group algebra $\mathbb{C}[S_n]$, which is simply a vector space with the $n!$ elements of $S_n$ indexing the basis elements. This is an $n!$ dimensional vector space which is also an algebra, which means we can multiply elements via the group operation; i.e., $e_ge_h = e_{gh}$. Now take a partition $\lambda \vdash n$ and fill it in a standard way. Now define the following elements of $\mathbb{C}[S_n]$. Let $R(\lambda)$ and $C(\lambda)$ be the set of permutations in $S_n$ which preserve the rows and columns of $\lambda$, respectively.

\[
a_\lambda = \sum_{g \in R(\lambda)} e_g \quad b_\lambda = \sum_{g \in C(\lambda)} (\text{sgn } g)e_g \quad c_\lambda = a_\lambda b_\lambda
\]

Then we can check that the image of $c_{(n)} = a_{(n)}$ in $V^\otimes n$ is indeed $S^nV$, and $c_{(11\ldots1)} = b_{(11\ldots1)}$ gives $\wedge^nV$. In general, $V^\otimes n$ breaks down into a direct sum of the spaces given by the images of these $c_\lambda$s which are indeed irreducible representations of $GL(V)$. These $c_\lambda$s, also known as **Young symmetrizers**, can be used to construct the irreducible representations of $S_n$ as well. They present a deep connection between $S_n$ and $GL(V)$ which is explained in more detail in Schur-Weyl duality.
Chapter 2
Basic representation theory of finite groups

2.1 Definitions

Let $V$ a complex vector space of dimension $n$. Then if there is a homomorphism $\rho : G \to GL(V)$, we say that $V$ is a representation of $G$. The idea is that $V$ is simply a space which a group acts on like a group of matrices. We will often abuse notation and write both $\rho(g)v$ and $gv$ to mean the same thing.

A map between two representations $V$ and $W$ of $G$ is a linear map $\phi$ that preserves the action of $G$. This means that $g\phi(v) = \phi(gv)$. These maps are also called $G$-module homomorphisms.

A subrepresentation $W$ of $V$ is a subspace of $V$ such that $gW \subseteq W \forall w \in W$. If $V$ has no subrepresentations other than 0 and itself, we say that it is irreducible. If it cannot be written as the direct sum of two subrepresentations, it is indecomposable. Note that irreducible implies indecomposable, and while these concepts aren’t always the same in general, they are the same for finite groups over a field of characteristic 0.

Given a representations $V$ and $W$, we can define the direct sum and tensor representations as expected: $g(v \oplus w) = gv \oplus gw$ and $g(v \otimes w) = gv \otimes gw$. We can also define the dual representation of $V^*$. Here we want to preserve the pairing, so as matrices we have $\rho^*(g) = \rho(g^{-1})^T$; understanding this is left as an exercise!

Finally, we can define the regular representation of $G$ by creating a vector space with a basis indexed by the elements of $G$, so that $ge_h = e_{gh}$.

2.2 Maschke’s theorem and Schur’s lemma

Maschke’s theorem says that every finite dimensional representation $V$ (over $\mathbb{C}$) of a finite group $G$ can be written as a direct sum of irreducible representations. Essentially, this just says that if
there is a subrepresentation $W$ of $V$, then we can find another subrepresentation $W'$ of $V$ such that $V = W \oplus W'$. Note that this isn’t true for all groups; e.g. consider the representation of $Z : a \rightarrow \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$ acting on $\mathbb{C}^2$.

**Theorem 2.2.1** (Maschke’s theorem). Every finite dimensional representation $V$ (over $\mathbb{C}$) of a finite group $G$ can be written as a direct sum of irreducible representations.

**Proof.** Let $W$ be a subrepresentation of $V$; we want to find a complementary subrepresentation. Take some Hermitian inner product $H_0$ on $V$ and define another Hermitian product $H$ on $V$ by setting $H(v, w) = \sum_{g \in G} H_0(gv, gw)$. $H$ has the advantage of being $G$-invariant. Then the orthogonal complement of $W$ will work.

OR: Say $V = W \oplus U$; we would be done if $U$ is $G$-invariant. Now let $\pi_0 : V \rightarrow W$ be the projection map and let $\pi(v) = \frac{1}{|G|} \sum_{g \in G} g\pi_0(g^{-1}v))$. Note that this is a projection onto $W$. Then the kernel of this map is complementary to $W$ and is $G$-invariant.

**Theorem 2.2.2** (Schur’s lemma). If $V$ and $W$ are irreducible representations of $G$ and $\phi : V \rightarrow W$ is a $G$-module homomorphism, then $\phi$ is multiplication by a constant or $0$.

**Proof.** Note that the kernel and image of $\phi$ are $G$-invariant. Then since $V$ and $W$ are irreducible, $\phi$ is either an isomorphism or $0$. If it is an isomorphism, then it has an eigenvalue $\lambda$, and then applying what we just proved to the map $\phi - \lambda I$ gives that $\phi = \lambda I$.

Another way to phrase Schur’s Lemma is the following: If $V$ and $W$ are irreducible representations, then $\dim \text{Hom}(V, W)^G = 1$ if $V \cong W$ as $G$-modules and $0$ otherwise.

### 2.3 Character theory

Character theory gives a very powerful tool for studying representations.

**Definition 2.3.1.** The **character** of a representation $V$ is a function on $G$: $\chi_V : G \rightarrow \mathbb{C}$ with $\chi(g) = \text{Tr}(\rho(g))$.

A character is a class function on $G$; i.e., it is constant across conjugacy classes of $G$. Indeed, because trace is independent of basis, we have $\chi_V(h^{-1}gh) = \text{Tr}(\rho(h^{-1}gh)) = \text{Tr}(\rho(h)^{-1}\rho(g)\rho(h)) = \text{Tr}(\rho(g)) = \chi_V(g)$.

If we consider all the class functions of $G$, we see that they form a vector space with cardinality equal to the number of conjugacy classes of $G$. Now the amazing fact is that, if we enumerate the irreducible representations of $G$, there are precisely this many of them, and they in fact form a basis for these class functions. In fact, they are an orthogonal basis with respect to the inner product $(\chi_1, \chi_2) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\chi_2(g)$. The main idea we will use is the same notion of averaging as we saw in the proof of Maschke’s theorem, and we will obtain the inner product by noting its connection to the representation of $\text{Hom}(V, W) = V^* \otimes W$.

We will split this theorem into two parts. First we will use a lemma.

Define $V^G$ to be the elements $v \in V$ such that $gv = v$. In other words, it is the collection of trivial subrepresentations of $V$. Now define $\phi : V \rightarrow V^G$ by $\phi(g) = \frac{1}{|G|} \sum_{g \in G} gv$. 8
Lemma 2.3.2. \( \frac{1}{|G|} \sum_{g \in G} \chi_V(g) = \text{Tr}(\phi) = \text{dim}(V^G) \).

Proof. The first equality holds from the linearity of trace. The second equality follows from taking an appropriate basis of \( V \). \qed

Theorem 2.3.3. If \( V \) and \( W \) are irreducible representations of \( G \), then \( (\chi_V, \chi_W) = 1 \) if \( V \cong W \) as \( G \)-modules and 0 otherwise.

Proof. Note that we have \( (\chi_V, \chi_W) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\chi_W(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V \otimes \chi_W(g) \). By the lemma above, we have that this is equal to \( \text{dim}((V^* \otimes W)^G) \). Now an element of \( A \in V^* \otimes W \) satisfies \( gA = A \) if and only if \( g(A(g^{-1}(v))) = g(v) \); i.e., it is a \( G \)-module homomorphism. By Schur’s lemma, this is 1 if \( V \cong W \) as \( G \)-modules and 0 otherwise. \qed

This implies that the number of irreducible representations of \( G \) is less than or equal to the number of its conjugacy classes. In fact, these numbers are equal. To prove this, we consider the various endomorphisms given by \( g \) again, but instead of averaging them, we consider which linear combinations are maps of \( G \)-modules.

Theorem 2.3.4. Let \( \alpha : G \to \mathbb{C} \) be a function and consider the linear map \( \phi_{\alpha,V}(v) = \sum_{g \in G} \alpha(g) \cdot g(v) \). This is a \( G \)-module homomorphism for all \( V \) iff \( \alpha \) is a class function.

Proof. If \( \alpha \) is a class function, it is easy to see by definition that \( \phi_{\alpha,V} \) is a \( G \)-module homomorphism. If it is not, then it won’t be a \( G \)-module homomorphism for the regular representation. (Exercise!) \qed

This allows us to conclude:

Theorem 2.3.5. The characters of the irreducible representations of \( G \) form an orthonormal basis for the class functions on \( G \).

Proof. It remains to show that the only class function orthogonal to all \( \chi_V \) for \( V \) irreducible is 0. Take such a function \( \alpha \); then by the theorem above we have that \( \phi_{\pi,V} \) is a \( G \)-module homomorphism and thus by Schur’s lemma it is multiplication by a constant. Taking the trace, we see that in fact, \( \phi_{\pi,V} = 0 \). This means that \( \sum_{g \in G} \pi(g)(g) \) is 0 on every representation of \( G \); taking the regular representation shows that this forces \( \alpha = 0 \). \qed

There are many consequences of this left as exercises.
Chapter 3

Representation theory of $S_n$

3.1 Introduction

The structure behind the representations of the symmetric group is very interesting. They have many connections with other areas of math, some of which we will try to explore. As usual, we are interested in understanding what the irreducible representations look like. By the general theory, these are in bijective correspondence with the conjugacy classes of $S_n$. Because the conjugacy class of an element in $S_n$ only depends on its cycle type, these are indexed by the partitions of $n$. So for each partition $\lambda \vdash n$, we have an irreducible representation of $S_n$.

We will discuss two methods for constructing the irreducible representations of $S_n$. The first involves tabloids and has the advantage of being very explicit and straightforward to work with. The second uses Young symmetrizers and can be extended to understand Schur functors and Schur-Weyl duality.

3.2 Tabloid approach

3.2.1 Tabloids and the permutation module

A tableau is a Young diagram of some partition $\lambda \vdash n$ filled in with the numbers $1, 2, \ldots, n$. Define a tabloid to be an equivalence class of tableau, where two tableau are considered equal if they have the same row sets. For instance, the equivalence class \{ \scalebox{0.8}{\begin{tabular}{c} 1 2 \\ 3 \end{tabular}} , \scalebox{0.8}{\begin{tabular}{c} 2 1 \\ 3 \end{tabular}} \} is a tabloid. Note that the order of the rows does matter; in particular, \[ \scalebox{0.8}{\begin{tabular}{c} 1 2 \\ 3 4 \end{tabular}} \] \[ \scalebox{0.8}{\begin{tabular}{c} 3 4 \\ 1 2 \end{tabular}} \] under this equivalence relation.

The set of tabloids of a given type (i.e., partition) carries a natural $S_n$ action, and are thus a representation of $S_n$. If $\lambda \vdash n$ is a partition of $n$, then we call $M^\lambda$ the permutation module of type $\lambda$. To distinguish tableau from tabloids, we will write the tabloid coming from a tableau $t$ as \{ $t$ \}. The way $S_n$ acts on the tabloids is perfectly natural: it simply acts on each of the numbers written inside the tableau. Note that this means that the stabilizer of $M^\lambda$ is the so-called Young subgroup,
consisting of the product $S_{\lambda_1} \times \cdots \times S_{\lambda_d}$ where each part consists of the permutations of the numbers in each row of the tabloid. We denote this Young subgroup, also known as the row subgroup, by $R_t$. Similarly, we can define the column subgroup $C_t$ to be the subgroup of permutations which preserve all the columns.

Now given any tableau $t$, we can define an element in $\kappa_t \in \mathbb{C}[S_n]$ as follows: $\kappa_t = \sum_{w \in C_t} \text{sgn}(w)w$. We can clearly define this map so that the image is made of tabloids rather than tableau. Now we define a polytabloid to be an element in $M^\mu$ of the form $\kappa_t(t)$, where the image is taken to be made of tabloids. That is, the polytabloid $e_t = \{\kappa(t)\}$. For instance, if $t = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$, we have

$$e_t = \{ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} - \begin{array}{c} 3 \\ 2 \\ 1 \\ 4 \end{array} - \begin{array}{c} 1 \\ 4 \\ 3 \\ 2 \end{array} + \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} \}.$$  

Note that two tableau which are in the same tabloid class can give different polytabloids.

### 3.2.2 Specht modules

Now we can define the Specht modules $S^\lambda$, which are indeed the irreducible representations of $S_n$.

**Definition 3.2.1.** The Specht module $S^\lambda$ is the $\mathbb{C}[S_n]$-module generated by $e_t$ for any polytabloid of type $t$.

Of course we need to show that these are well-defined, which amounts to showing that whatever polytabloid we choose of some type $t$, we will get the same module. But it is easy to check that the action of $S_n$ commutes with this map $t \rightarrow e_t$, so this is true.

Before working out the proof that these are what we’re looking for, let’s work through some examples.

For $n = 3$, we have that $S^{(3)}$ is the trivial representation. $S^{(111)}$ is also one-dimensional, with basis $e_{123} - e_{213} - e_{132} - e_{321} + e_{231} + e_{312}$. It is the sign representation. Finally, consider $S^{(21)}$.

For convenience, let $1 = \{3\}, 2 = \{2\}, 1 = \{1\}$. Then each tabloid (up to sign) is of the form $2 - 1, 3 - 1, 3 - 2$. This is the standard representation of dimension 2.

### 3.2.3 Proofs

### 3.3 Young symmetrizer approach
Chapter 4

Schur-Weyl duality
Chapter 5

Problems

5.1 Tensor products

The definition of tensor product for modules is perfectly analogous to the second definition we gave for vector spaces. If \( M, N \) are \( R \)-modules, then their tensor product is denoted \( M \otimes_R N \).

1. Using the universal property, show that tensor products are unique up to isomorphism.

2. Compute \( \mathbb{Z}_n \otimes \mathbb{Z}_n \).

3. Show that \( \mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_{(m,n)} \).

4. If \( V \) is finite dimensional, show that \( \text{Hom}(V,W) \cong V^* \otimes W \). In fact, this isomorphism should not depend on the choice of basis for \( V \).

5. Let \( V \) be of dimension \( n \) and let \( A : V \to V \) be a linear map. Show that \( \wedge^n A = \det(A)\text{Id} \).

6. Show that \( S^n V \) and \( \wedge^n V \) are indeed irreducible representations of \( GL(V) \). At least try it for \( n = 2 \).

5.2 Basic representation theory

1. Using the map \( \mathbb{Z} : a \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), show that Maschke’s theorem fails for \( \mathbb{Z} \).

2. Show that a representation \( V \) of \( G \) is irreducible iff \( Gv \) spans \( V \) for all \( v \in V \).

3. Using the above exercise, show that we can find a copy of every irreducible representation inside the regular representation.

4. Prove Maschke’s theorem and Schur’s lemma. (i.e., go through the proofs given above carefully!)

5. By considering a linear map as an element of \( V^* \otimes V \), show that trace is independent of basis.

6. We want the dual representation \( \rho^* \) to satisfy \( \langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle \). Show that, as a matrix defining a linear transformation \( V^* \to V^* \), we want to define \( \rho^*(g) = \rho(g^{-1})^\tau \).
7. Recall that \( \text{Hom}(V, W) \cong V^* \otimes W \). Show that, as a representation of \( G \), if \( A \in \text{Hom}(V, W) \), then \( (gA)(v) = g(A(g^{-1}(v))) \).

8. Using Schur’s lemma, show that all irreducible representations of an abelian group are 1-dimensional. (Hint: construct a \( G \)-module homomorphism from \( V \) to itself for every element \( g \in G \).

9. Consider the regular representation of \( C_n \). Show explicitly that it decomposes into a sum of 1-dimensional representations by finding those subspaces.

10. Show that, in the proof of Theorem 2.3.4, the regular representation is indeed a counterexample if \( \alpha \) is not a class function.

The following three problems are consequences of Theorem 2.3.5.

11. Show that an irreducible representation \( V \) of \( G \) appears in the regular representation \( \dim V \) times.

12. Show that the order of \( G \) is equal to the sum of the squares of the dimensions of its irreducible representations.

13. Show that the columns of the character table are orthogonal. (Hint: it is straight linear algebra using the fact that the rows are orthogonal.)
Part II

Representation theory of $S_n$ and $GL_n$
Chapter 6

The Flag Manifold

In this chapter, we discuss the flag manifold. This is a prototype for the generalized flag variety, in which the main ideas (Borel subgroups, Bruhat decomposition, etc.) can be worked out for any algebraic group, which we take to be $GL_n\mathbb{C}$ in this case.

6.1 Construction

Let $V$ be a vector space over $\mathbb{C}$ of dimension $n$. Then a (complete) flag of $F$ is a collection of subspaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V,$$

where $\dim(V_i) = i$. We can identify every flag with an element $g \in G = GL_n(\mathbb{C})$ by picking the $i$th column to be an element of $V_i \setminus V_{i-1}$. However, it is evident that such a choice of $g$ is not unique.

Let $B$ be the set of upper-triangular matrices in $GL_n(\mathbb{C})$. They form a subgroup of $GL_n(\mathbb{C})$. $B$ is called a Borel subgroup of $GL_n\mathbb{C}$. Then there is an isomorphism

$$\text{Flags} \leftrightarrow GL_n(\mathbb{C})/B,$$

where $GL_n(\mathbb{C})/B$ is a coset space. In other words, we can identify each flag with a coset $gB$ for some $g \in G$.

The set of flags have both a manifold and variety structure. Neither of these is trivial to prove, but the first is easier to understand and the second will be something we develop carefully in further chapters. Note that $GL_n(\mathbb{C})$ is a manifold of dimension $n^2$ and that $B$ is a submanifold of dimension $\frac{n(n+1)}{2}$. When $B$ is also a subgroup of $G$, then it can be shown that the coset space $gB$ is also a manifold with dimension equal to their difference. So here, the dimension of the flag manifold is $\frac{n(n-1)}{2}$.

6.2 Bruhat decomposition

The Bruhat decomposition is a way of decomposing $G = GL_n(\mathbb{C})$ into double cosets, which can be used to decompose the set of flags as well.
Theorem 6.2.1.  
\[ G = \coprod_{w \in S_n} BwB. \]

Proof. We first show that every element \( g \in G \) can be represented in such a way. Note that it suffices to show that we can write \( gb_1 \) as \( b_2w \) for some \( b_1, b_2 \in B \) and \( w \in S_n \). So first we take \( b_1 \) to be the element that column reduces as much as possible; this will make it so that every column has a ‘lowest’ entry so that everything to the right in its row is 0. Then looking at these pivots as the entries of \( S_n \), we can always pick an appropriate \( b_2 \) to make sure everything above these pivots is what we want.

The proof that these double cosets are disjoint is left as an exercise. \( \square \)

Example 6.2.2. 
\[ g = \begin{bmatrix} 2 & 5 & 0 \\ 3 & 6 & 0 \\ 1 & 2 & 1 \end{bmatrix} \rightarrow gb_1 = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \]

This gives a natural way to decompose the flag variety: for each \( w \in S_n \), let \( C_w := (Bw)B \). \( C_w \) is called a Schubert cell, and their closures are called Schubert varieties. They consist of a set of flags.

A natural question to ask is: what is the dimension of these Schubert cells? To answer this, write every element of a Schubert cell in the canonical form with the pivots. Then notice that every element of the Schubert cell is simply the number of free spots that aren’t constrained to be 0; e.g. in our above example, \( w = 312 \) and the free spots for this Schubert cell are shown below:

\[ \begin{bmatrix} * & 1 & 0 \\ * & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \]

Thus dim(\( C_{312} \)) = 2. This illustrates the following theorem and its proof:

**Theorem 6.2.3.** \( \dim(C_w) = |\text{inv}(w)|. \)

6.3 The Bruhat order

The Bruhat order is a partial ordering on \( S_n \) (or more generally, any Coxeter group) that tells us how these Schubert cells interact with each other. First, define the length function \( l(w) \) to be the number of inversions of \( w \). It’s a number between 0 and \( \frac{n(n-1)}{2} \) inclusive.

**Definition 6.3.1.** The (strong) Bruhat order is defined as the transitive and reflexive closure of the relations \( w_1 > w_2 \) if \( w_1 = (i,j)w_2 \) and \( l(w_1) = l(w_2) + 1. \)

**Example 6.3.2.**

Now consider the closure of the Schubert cells \( C_w \). Unfortunately, to make this precise we would have to go through how its topology is defined as a projective variety, which will take some time.
But intuitively, we can see that as some of the $*$s go to infinity, some of these Schubert cells limit to each other. For instance,

$$\begin{bmatrix} \infty & 1 & 0 \\ \infty & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} * & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This shows that $C_{213} \in C_{312}$. In general, we have the following nice theorem:

**Theorem 6.3.3.**

$$\overline{C_w} = \coprod_{v \leq w} C_v,$$

*where the ordering is the Bruhat order.*
Chapter 7

Problems

1. Find $|GL_n(F_q)|$.

2. Show that the distinct double cosets $BwB$ are disjoint.

3. Prove Theorem 6.2.3: $\dim(C_w) = |\text{inv}(w)|$.

4. Show that the number of permutations with length $k$ is the same as the number of permutations with length $\frac{n(n-1)}{2} - k$.

5. A matrix is said to be monomial if each row and column has exactly one non-zero entry. Let $N$ be the subset of $G = GL_n(F)$ consisting of all monomial matrices. Show that $N \leq G$, that $T = B \cap N$ is the subgroup of $G$ consisting of all diagonal matrices, that $N = N_G(T)$, and that $N = T \rtimes W$. 

Bibliography

[1] Fulton and Harris, *Representation Theory*

[2] Alperin and Bell, *Groups and Representations*